Superconvergence of discontinuous Galerkin methods for scalar nonlinear conservation laws in one space dimension

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Outline

Why to use DG method?

For time-dependent nonlinear hyperbolic equations, the exact solution always develops discontinuities as time evolves.

Features of DG method

- High order accuracy: in obtaining arbitrary high order accuracy approximation to the exact solution within smooth regions
- High resolution: in producing sharp and non-oscillatory discontinuity transitions near discontinuous solutions, including shocks and contact discontinuities,

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Example 1: Burgers equation

$$\begin{cases} u_t + (u^2/2)_x = 0 \\ u(x,0) = 1/2 + \sin x \end{cases} \tag{1}$$

Accuracy test

Table: The numerical errors and orders when using P^2 polynomials of N cell at $T = 0.3$

N	L^1 error	Order	L^∞ error	Order
20	1.09E-04	–	9.09E-04	–
40	1.34E-05	3.03	1.48E-04	2.62
80	1.63E-06	3.04	2.07E-05	2.84
160	2.01E-07	3.02	2.78E-06	2.90
320	2.50E-08	3.01	3.61E-07	2.94
640	3.13E-09	3.00	4.60E-08	2.97
1280	3.91E-10	3.00	5.81E-09	2.99

Test with shocks

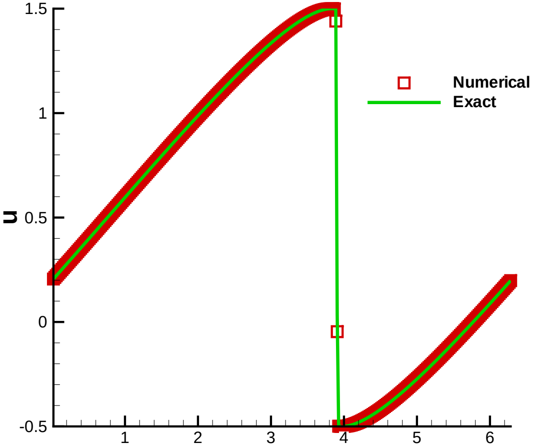

Figure: P^2 polynomials, $T = 1.5$, $N = 320$.

Example 2: Euler equation

Consider the Sod problem

$$\mathbf{u}_t + \mathbf{f(u)}_x = 0,$$

where

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}, \quad \mathbf{f(u)} = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix},$$

and

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2, \quad \gamma = 1.4, \quad x \in [-5, 5], \quad t = 2.$$

The initial condition is

$$(\rho(x,0),\ v(x,0),\ p(x,0)) = \begin{cases} (1,\ 0,\ 1), & \textit{if} \quad x \le 0 \\ (0.125,\ 0,\ 0.1), & \textit{if} \quad x > 0 \end{cases}$$

Density

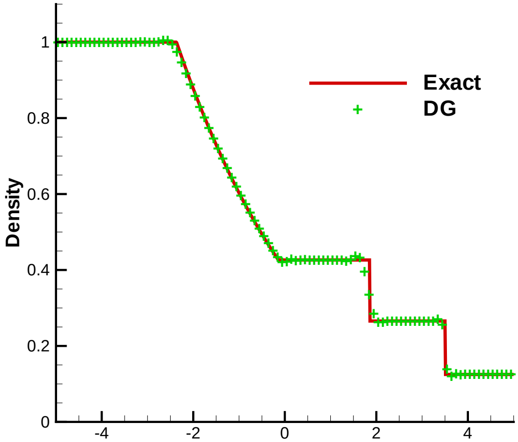

Figure: The computed density using P^1 polynomials of 100 cells at $T = 2$.

Pressure

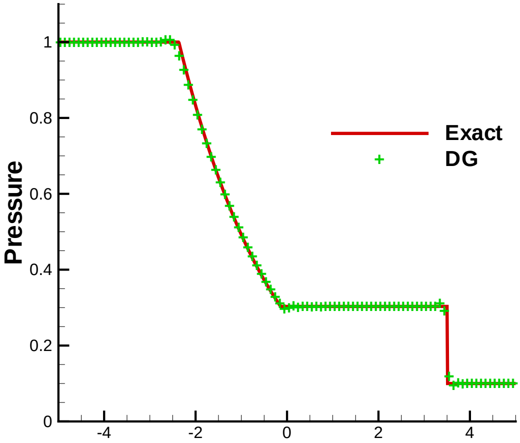

Figure: The computed pressure using P^1 polynomials of 100 cells at $T = 2$.

The design of DG method

Model equation

Consider the one-dimensional nonlinear conservation laws

$$u_t + f(u)_x = 0 \quad (2a)$$
$$u(x,0) = u_0(x) \quad (2b)$$

Step 1: Partition of the domain

Use the following mesh

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi$$

to cover the computational domain $I = (0, 2\pi)$, consisting of cells

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad j = 1, \ldots, N.$$

Cell centers and cell lengths

$$x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2 \quad \textit{and} \quad h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$$

and $h = \max_{1\le j\le N} h_j$.

The design of DG method

Step 2: Weak formulation

Multiply arbitrary smooth functions, v and w on the RHS of (5), then integrate on cell I_j and use integration by parts to obtain

$$\int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + f(u(x_{j+\frac{1}{2}}, t)) v(x_{j+\frac{1}{2}}) - f(u(x_{j-\frac{1}{2}}, t)) v(x_{j-\frac{1}{2}}) = 0 \quad (3a)$$

$$\int_{I_j} u(x,0) w dx - \int_{I_j} u_0(x) w dx = 0 \quad (3b)$$

The finite element space is

$$V_h^k = \{v \in L^2(I) : v|_{I_j} \in P^k(I_j), \;\; j = 1, \ldots, N\}$$

where $P^k(I_j)$ denotes the set of polynomials of degree up to k defined on the cell I_j.

The design of DG method

Step 3: DG scheme

Find the unique function $u_h(x,t) \in V_h^k$ and $u_h(x,0)$ such that

$$\int_{I_j} (u_h)_t v_h dx - \int_{I_j} f(u_h)(v_h)_x dx + \hat{f}_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} = 0 \quad (4a)$$

$$\int_{I_j} u_h(x,0) w_h dx - \int_{I_j} u_0(x) w_h dx = 0 \quad (4b)$$

holds for all $v_h, \ w_h \in V_h^k$ and $j = 1, \ldots, N$.

Monotone numerical flux

- $\hat{f}_{j+\frac{1}{2}} = \hat{f}\left((u_h)^-_{j+\frac{1}{2}}, (u_h)^+_{j+\frac{1}{2}}\right)$
- Consistency
- Lipschitz continuity
- Monotonicity

Runge–Kutta DG method

Semi-discrete scheme

After using the DG method, we get

$$u_t = L(u, t)$$

Time discretization

Adopt the explicit third-order TVD Runge–Kutta time discretization [Shu & Osher, JCP, 88']

$$u^{(1)} = u^n + \Delta t L(u^n, t^n)$$

$$u^{(2)} = \frac{3}{4}u^n + \frac{1}{4}\left(u^{(1)} + \Delta t L(u^{(1)}, t^n + \Delta t)\right)$$

$$u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}\left(u^{(2)} + \Delta t L(u^{(2)}, t^n + \frac{1}{2}\Delta t)\right)$$

Local DG (LDG) method

Introduction of the method

The LDG was first proposed in the framework of second order convection diffusion equations [Cockburn & Shu, SINUM, 98']

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- Rewrite the equation into a first order system by introducing auxiliary variables
- Apply the DG method on the system

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Basic idea

- Rewrite the equation into a first order system by introducing auxiliary variables
- Apply the DG method on the system

Criteria of numerical fluxes

- Guarantee stability of the scheme
- Guarantee local solvability of all the auxiliary variables

Advantages of DG and LDG methods

- Arbitrary high order accuracy theoretically
- Flexible to $h-p$ adaptivity
- Extremely local date communications
- Capacity in handing complicated geometry and boundary conditions
- Provable nonlinear L^2 stability: [Jiang & Shu, Math. Comp., 94']
- High parallel efficiency

Types of superconvergence

I : Negative norm, post-processing

- $\|u - K_H^{\nu,l} \star u_h\| \leq \frac{H^\nu}{\nu!} C_1 \|u\|_{H^\nu} + C_2 \sum_{|\alpha| \leq l} \|\partial_H^\alpha (u - u_h)\|_{H^{-l}}$
- $\|v\|_{-l} = \sup_{\phi \in C_0^\infty} \frac{(v,\phi)}{\|\phi\|_{H^l}}$

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- $\|P_h u - u_h\| \leq Ch^{k+1+\alpha}$
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- α could be $\frac{1}{2}$ or 1

III : At Radau points, and cell averages

- $\left(\frac{1}{N} \sum_{j=1}^{N} |(u - u_h)(x_j)|^2\right)^{\frac{1}{2}} \leq Ch^{k+2}$
- $\|\overline{u - u_h}\| \leq Ch^{k+2}$

Approaches

Fourier type: quantitative analysis

- Uniform meshes
- Periodic boundary conditions
- Piecewise linear elements

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Fourier type: quantitative analysis

- Uniform meshes
- Periodic boundary conditions
- Piecewise linear elements

Finite element type: qualitative analysis

- Arbitrary nonuniform regular meshes
- Periodic boundary conditions and initial-boundary value problems
- Arbitrary piecewise polynomials of degree k

Some superconvergence results

Linear hyperbolic equations

- Negative norm, post-processing
  - $(2k+1)th$, [Cockburn, Luskin, Shu & Süli, Math. Comp., 03'], [Ryan, Shu & Atkins, SISC, 05'], [Mirzaee, Ji, Ryan & Kirby, SINUM, 11']

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- Towards special projection of the exact solution
  - Fourier type, $(k+\frac{3}{2})th$: [Cheng & Shu, JCP, 08']
  - Finite element type, $(k+2)th$: [Yang & Shu, SINUM, 12']

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- At Radau points, and cell averages
  - Fourier type: $(k+2)th$ at Radau points and $(2k+1)th$ at downwind point [Adjerid et al., CMAME, 02', steady-state], [Zhong & Shu, CMAME, 11']
  - Finite element type
    - $(k+2)th$ at Radau points, cell averages: [Yang & Shu, SINUM, 12']
    - Additional $(2k+1)th$ at downwind point, cell averages and pointwise $(k+1)th$ derivative superconvergence: [Cao, Zhang & Zou, SINUM, submitted]

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Some superconvergence results

Higher order PDEs

- Towards special projection of the exact solution
  - Finite element type
    - linearized KdV equations, $(k+\frac{3}{2})th$: [Hufford & Xing, JCAM, 14']
    - linear fourth-order equations, $(k+\frac{3}{2})th$: [Meng, Shu & Wu, IMANUM, 12']

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Nonlinear hyperbolic equations

- Negative norm, post-processing
  - $(2k+1)th$, [Ji, Xu & Ryan, JSC, 13']
- At Radau points
  - $(k+2)th$ at Radau points and $(2k+1)th$ at downwind point: [Adjerid & Massey, CMAME, 06', steady-state]

Problem

We consider the discontinuous Galerkin (DG) method applied to one-dimensional scalar conservation laws

$$u_t + f(u)_x = g(x,t), \quad (5a)$$
$$u(x,0) = u_0(x), \quad (5b)$$

here $g(x,t)$ and $u_0(x)$ are smooth functions and assume that $f(u) \in C^3$.

Our goal

To study the superconvergence (towards special projection of the exact solution) of the DG method for nonlinear hyperbolic conservation laws

Notation

$I = (0, 2\pi)$, $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, where

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi.$$

$x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$, $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$: The cell center and cell length;

$p^-_{j+\frac{1}{2}}$ and $p^+_{j+\frac{1}{2}}$: the left and right limit of p at $x_{j+\frac{1}{2}}$;

$[\![p]\!] = p^+ - p^-$ and $\{\!\{p\}\!\} = \frac{1}{2}(p^+ + p^-)$: the jump and the mean of p at each element boundary point;

$V_h \equiv V_h^k = \{v \in L^2(0, 2\pi) : v|_{I_j} \in P^k(I_j), \quad j = 1, \cdots, N\}$: finite element space, where $P^k(I_j)$ denotes the set of polynomials of degree up to $k \geq 1$ defined on the cell I_j.

DG scheme

Find the unique function $u_h = u_h(t) \in V_h$ such that

$$\int_{I_j} (u_h)_t v_h dx - \int_{I_j} f(u_h)(v_h)_x dx + \hat{f}_{j+\frac{1}{2}}(v_h)^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}(v_h)^+_{j-\frac{1}{2}} = \int_{I_j} g(x,t) v_h dx \qquad (6)$$

holds for all $v_h \in V_h$ and all $j = 1, \cdots, N$.

Numerical flux $\hat{f}_{j+\frac{1}{2}}$ is chosen to be an upwind flux to achieve superconvergence.

Functionals related to the L^2 norm

$$\mathcal{B}_j^-(\mathbb{F}) = \int_{I_j} \mathbb{F}(x) \frac{x - x_{j-1/2}}{h_j} \frac{d}{dx}\left(\mathbb{F}(x)\frac{x - x_j}{h_j}\right) dx,$$

$$\mathcal{B}_j^+(\mathbb{F}) = \int_{I_j} \mathbb{F}(x) \frac{x - x_{j+1/2}}{h_j} \frac{d}{dx}\left(\mathbb{F}(x)\frac{x - x_j}{h_j}\right) dx.$$

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Lemma

For any function $\mathbb{F}(x) \in C^1$ on I_j, we have

$$\mathcal{B}_j^-(\mathbb{F}) = \frac{1}{4h_j}\int_{I_j} \mathbb{F}^2(x)dx + \frac{\mathbb{F}^2(x_{j+1/2})}{4}, \tag{7}$$

$$\mathcal{B}_j^+(\mathbb{F}) = -\frac{1}{4h_j}\int_{I_j} \mathbb{F}^2(x)dx - \frac{\mathbb{F}^2(x_{j-1/2})}{4}. \tag{8}$$

Projections and interpolation properties

- L^2 projection

$$\int_{I_j} (P_h q(x) - q(x)) v_h dx = 0, \quad \forall v_h \in V_h.$$

- Gauss-Radau projections $P_h^{\pm}$ into V_h

$$\int_{I_j} (P_h^+ q(x) - q(x)) v_h dx = 0, \ \forall v_h \in P^{k-1}, \quad (P_h^+ q)^+_{j-\frac{1}{2}} = q(x^+_{j-\frac{1}{2}}); \tag{9}$$

$$\int_{I_j} (P_h^- q(x) - q(x)) v_h dx = 0, \ \forall v_h \in P^{k-1}, \quad (P_h^- q)^-_{j+\frac{1}{2}} = q(x^-_{j+\frac{1}{2}}). \tag{10}$$

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1. Orthogonality property for polynomials of degree up to $k-1$
2. Exact collocation at one of the boundary points

Projections and interpolation properties

Denote by $\eta = q(x) - \mathbb{Q}_h q(x)$ ($\mathbb{Q}_h = P_h$, or $P_h^{\pm}$) the projection error, then by Bramble-Hilbert Lemma and scaling argument, we have

$$\|\eta\| + h\|\eta_x\| + h^{1/2}\|\eta\|_{\Gamma_h} \leq Ch^{k+1}. \tag{11a}$$

Here and below, an unmarked norm $\|\cdot\|$ is the usual L^2 norm defined on the interval I, and

$$\|\eta\|_{\Gamma_h}^2 = \sum_{j=1}^{N} \left(\left(\eta_{j+1/2}^{+}\right)^2 + \left(\eta_{j+1/2}^{-}\right)^2 \right).$$

We also have

$$\|\eta\|_{\infty} \leq Ch^{k+\frac{1}{2}} \tag{11b}$$

The property (11b) is important for the a priori assumption.

Inverse properties

For any $p_h \in V_h$, there exists a positive constant C independent of p_h and h, such that

$$\text{(i)} \quad \|\partial_x p_h\| \le Ch^{-1}\|p_h\|;$$

$$\text{(ii)} \quad \|p_h\|_{\Gamma_h} \le Ch^{-1/2}\|p_h\|;$$

$$\text{(iii)} \quad \|p_h\|_\infty \le Ch^{-1/2}\|p_h\|.$$

$e = u - u_h$, $\eta = u - \mathbb{Q}_h u$ be the projection error, $\xi = \mathbb{Q}_h u - u_h$.
For any $t \in [0, T]$ and $x \in I$, if $f'(u(x,t)) > 0$, we choose $\mathbb{Q}_h = P_h^-$, if $f'(u(x,t)) < 0$, we take $\mathbb{Q}_h = P_h^+$.

Theorem

Let u be the exact solution of the problem (5), which is assumed to be sufficiently smooth, and assume that $f \in C^3$ and $|f'(u)|$ is lower bounded uniformly by any positive constant. Let u_h be the numerical solution of (7) with initial condition $u_h(\cdot, 0) = \mathbb{Q}_h u_0$ when the upwind flux is used. If the finite element space V_h^k $(k \geq 1)$ is used then for small enough h there holds the following error estimate

$$\|\xi(\cdot, t)\| \leq Ch^{k+3/2} \quad \forall t \in [0, T], \tag{12}$$

where C depends on the exact solution u, the final time T and the maximum of $|f^{(m)}|$ $(m = 1, 2, 3)$, but is independent of h.

We will only consider the case $f'(u(x,t)) \geq \delta > 0 \; \forall (x,t) \in I \times [0,T]$, the case of $f'(u(x,t)) \leq -\delta < 0$ is similar.

Choose $\hat{f} = f(u_h^-)$ on each cell interface and $\mathbb{Q}_h = P_h^-$ on each cell element, the initial condition is chosen as $u_h(\cdot, 0) = P_h^- u_0$.

The proofs are divided into **FIVE** steps as follows.

An important inequality of ξ

- Error equation:

$$\int_I e_t v_h dx = \sum_{j=1}^{N} \int_{I_j} (f(u) - f(u_h))(v_h)_x dx + \sum_{j=1}^{N} ((f(u) - f(u_h^-))[\![v_h]\!])_{j+\frac{1}{2}}$$

for all $v_h \in V_h$.

- Take $v_h = \xi$ and define $\xi = r_j + \mathbb{S}_j(x)(x - x_j)/h_j$ on each cell I_j, with $r_j = \xi(x_j)$ being a constant and $\mathbb{S}_j(x) \in P^{k-1}(I_j)$.
- We get the following inequality involving ξ

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 \le (\mathcal{C}(e) + C_\star h^{-3}\|e\|_\infty^2)\|\xi\|^2 + C_\star h^{k+1}\|\mathbb{S}\| + Ch^{2k+3}, \quad (13)$$

where $\mathcal{C}(e) = C + C_\star h^{-1}\|e\|_\infty$.

The a priori assumption

To deal with the nonlinearity of the flux $f(u)$ we shall make an a priori assumption that, for small enough h, there holds

$$\|\xi\| = \|\mathbb{Q}_h u - u_h\| \le h^2. \tag{14}$$

Later we will justify this a priori assumption (14) for piecewise polynomials of degree $k \ge 1$.

Corollary

Suppose that the interpolation property (11b) *is satisfied, then the a priori assumption* (14) *implies that*

$$\|e\|_\infty \le Ch^{\frac{3}{2}} \quad \text{and} \quad \|\xi\|_\infty \le Ch^{\frac{3}{2}}. \tag{15}$$

Proof. This follows from the inverse property (iii), the interpolation property (11b) and triangle inequality.

The a priori assumption

Under this a priori assumption, we can first get a crude bound for ξ, which is used to derive a sharp bound for e_t.

Corollary

If the a priori assumption (14) *holds, we have the following error estimates*

$$\|e\| \le Ch^{k+1} \quad \text{and} \quad \|\xi\| \le Ch^{k+1}. \tag{16}$$

Remark

This result can be viewed as a straightforward consequence of the fully discrete DG method for solving conservation laws, see e.g., [Zhang & Shu, SINUM, 04' and 10'].

Estimate of $\mathbb{S}$

Lemma

Under the same conditions as in Theorem 2, if, in addition, the a priori assumption (14) *holds, we have*

$$\|\mathbb{S}\| \le Ch\|e_t\| + Ch^{k+2}, \tag{17}$$

for any $t \in [0, T]$, where the positive constant C is independent of h and the approximate solution u_h.

Step 4

Estimate of e_t

Lemma

Under the same conditions as in Theorem 2, if, in addition, the a priori assumption (14) *holds, we have*

$$\|e_t\| \le Ch^{k+1} + C_\star h^{-\frac{1}{2}} \sqrt{\int_0^t \|\xi(s)\|^2 ds}, \tag{18}$$

for any $t \in [0, T]$*.*

Final estimate of ξ

Collecting all the above results, employing (15) implied by the a priori assumption (14) and by virtue of Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|^2 \le C_1\|\xi(t)\|^2 + C_2\int_0^t \|\xi(s)\|^2 ds + C_3 h^{2k+3}. \qquad (19)$$

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Note that there holds the following identity

$$\frac{d}{dt}\int_0^t \|\xi(s)\|^2 ds = \|\xi(t)\|^2. \tag{20}$$

Step 4

Final estimate of ξ

Collecting all the above results, employing (15) implied by the a priori assumption (14) and by virtue of Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|^2 \le C_1\|\xi(t)\|^2 + C_2\int_0^t \|\xi(s)\|^2 ds + C_3 h^{2k+3}. \tag{19}$$

Note that there holds the following identity

$$\frac{d}{dt}\int_0^t \|\xi(s)\|^2 ds = \|\xi(t)\|^2. \tag{20}$$

Adding twice of (19) and (20) up, we arrive at

$$\frac{d}{dt}\left(\|\xi(t)\|^2 + \int_0^t \|\xi(s)\|^2 ds\right) \le C_0\left(\|\xi(t)\|^2 + \int_0^t \|\xi(s)\|^2 ds\right) + Ch^{2k+3},$$

where $C_0 = \max(2C_1 + 1, 2C_2)$ and $C = 2C_3$ are positive constants independent of h.

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where $C_0 = \max(2C_1 + 1, 2C_2)$ and $C = 2C_3$ are positive constants independent of h. By Gronwall's inequality, we get

$$\|\xi(\cdot, t)\| \leq Ch^{k+3/2}. \tag{21}$$

Step 5

Justification of the a priori assumption

First of all, the a priori assumption is satisfied at $t = 0$ since $\xi(\cdot, 0) = 0$. For piecewise polynomials of degree k ($k \geq 1$), one can choose h small enough such that $Ch^{k+3/2} < \frac{1}{2}h^2$, where C is a constant in (12) determined by the final time T.

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Define $t^\star = \sup\{s \leq T : \|\mathbb{Q}_h u(t) - u_h(t)\| \leq h^2, \forall t \in [0,\ s]\}$, then we have $\|\mathbb{Q}_h u(t^\star) - u_h(t^\star)\| = h^2$ by continuity if $t^\star < T$. However, our main result (21) implies that $\|\mathbb{Q}_h u(t^\star) - u_h(t^\star)\| \leq Ch^{k+3/2} < \frac{1}{2}h^2$, which is a contradiction. Therefore, there always holds $t^\star = T$, and thus the a priori assumption (14) is justified.

Numerical examples

Time discretization: five stage, fourth order SSP Runge-Kutta method

CFL condition: $\Delta t = CFL\ h^2$. Initial condition: L^2 projection.

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Example 1

First we consider the following equation

$$\begin{cases} u_t + (u^3/3 + u)_x = g(x,t), \\ u(x,0) = \cos(x) \\ u(0,t) = u(2\pi, t) \end{cases} \tag{22}$$

where $g(x,t)$ is given by

$$g(x,t) = -(2 + \cos^2(x+t))\sin(x+t).$$

The exact solution is

$$u(x,t) = \cos(x+t).$$

Table: The errors ξ and e for Example 1 when using P^1 polynomials on a uniform mesh of N cells. $CFL = 0.5$.

P^1	N	$T=1$		$T=50$		$T=500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	20	2.10E-04	–	1.84E-04	–	2.45E-04	–
	40	2.65E-05	2.99	2.73E-05	2.76	3.90E-05	2.65
	80	3.31E-06	3.00	3.65E-06	2.90	5.10E-06	2.93
	160	4.14E-07	3.00	4.61E-07	2.98	6.53E-07	2.97
	320	5.17E-08	3.00	5.77E-08	3.00	8.21E-08	2.99
e	20	4.26E-03	–	4.26E-03	–	4.24E-03	–
	40	1.06E-03	2.00	1.06E-03	2.00	1.06E-03	2.00
	80	2.65E-04	2.00	2.66E-04	2.00	2.65E-04	2.00
	160	6.64E-05	2.00	6.64E-05	2.00	6.64E-05	2.00
	320	1.66E-05	2.00	1.66E-05	2.00	1.66E-05	2.00

Table: The errors ξ and e for Example 1 when using P^1 polynomials on a nonuniform mesh of N cells. $CFL = 0.5$.

P^1	N	$T=1$		$T=50$		$T=500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	20	5.86E-04	–	6.46E-04	–	6.21E-04	–
	40	6.19E-05	3.24	5.86E-05	3.46	5.43E-05	3.51
	80	1.18E-05	2.39	7.71E-06	2.93	8.03E-06	2.76
	160	2.30E-06	2.37	7.81E-07	3.30	9.81E-07	3.03
	320	4.65E-07	2.30	1.14E-07	2.78	1.21E-07	3.02
e	20	5.50E-03	–	4.98E-03	–	5.37E-03	–
	40	1.30E-03	2.08	1.23E-03	2.01	1.28E-03	2.07
	80	3.55E-04	1.88	3.52E-04	1.81	3.52E-04	1.86
	160	8.73E-05	2.02	8.32E-05	2.08	8.36E-05	2.07
	320	2.13E-05	2.03	2.10E-05	1.99	2.15E-05	1.96

Table: The errors ξ and e for Example 1 when using P^2 polynomials on a uniform mesh of N cells. $CFL = 0.5$.

P^2	N	$T=1$		$T=50$		$T=500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	20	6.35E-06	–	6.70E-06	–	6.69E-06	–
	40	4.12E-07	3.94	4.13E-07	4.02	4.13E-07	4.02
	80	2.57E-08	4.00	2.57E-08	4.00	2.57E-08	4.00
	160	1.61E-09	4.00	1.61E-09	4.00	1.61E-09	4.00
	320	1.00E-10	4.00	1.00E-10	4.00	1.01E-10	3.99
e	20	1.07E-04	–	1.07E-04	–	1.07E-04	–
	40	1.34E-05	3.00	1.34E-05	3.00	1.34E-05	3.00
	80	1.67E-06	3.00	1.67E-06	3.00	1.67E-06	3.00
	160	2.09E-07	3.00	2.09E-07	3.00	2.09E-07	3.00
	320	2.61E-08	3.00	2.61E-08	3.00	2.61E-08	3.00

Table: The errors ξ and e for Example 1 when using P^3 polynomials on a uniform mesh of N cells. $CFL = 0.1$.

P^3	N	$T = 10$		$T = 50$		$T = 500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	10	2.82E-06	–	1.81E-06	–	1.98E-06	–
	20	5.47E-08	5.69	5.67E-08	5.00	5.66E-08	5.13
	40	1.74E-09	4.97	1.74E-09	5.02	1.74E-09	5.02
	80	5.42E-11	5.00	5.42E-11	5.00	5.49E-11	4.99
e	10	3.31E-05	–	3.30E-05	–	3.30E-05	–
	20	2.07E-06	4.00	2.07E-06	4.00	2.07E-06	4.00
	40	1.29E-07	4.00	1.29E-07	4.00	1.29E-07	4.00
	80	8.07E-09	4.00	8.07E-09	4.00	8.07E-09	4.00

Example 2

In this example, we solve the following equation

$$\begin{cases} u_t + (u^3/3)_x = g(x,t), \\ u(x,0) = \cos(x) \\ u(0,t) = u(2\pi,t) \end{cases} \tag{23}$$

where $g(x,t)$ is given by

$$g(x,t) = -(1+\cos^2(x+t))\sin(x+t).$$

The exact solution is

$$u(x,t) = \cos(x+t).$$

Table: The errors ξ and e for Example 2 when using both P^1 and P^2 polynomials on a nonuniform mesh of N cells. $CFL = 0.5$. $T = 1$.

P^k	$k=1$				$k=2$			
N	ξ		e		ξ		e	
	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
40	2.28E-04	–	1.08E-03	–	4.29E-06	–	1.40E-05	–
80	4.52E-05	2.33	2.75E-04	1.98	3.25E-07	3.72	1.77E-06	2.98
160	7.95E-06	2.51	6.85E-05	2.01	2.24E-08	3.86	2.18E-07	3.02
320	1.49E-06	2.42	1.72E-05	1.99	1.90E-09	3.56	2.77E-08	2.98
640	2.63E-07	2.50	4.30E-06	2.00	1.66E-10	3.52	3.48E-09	2.99

Example 3

We consider the following Burgers equation

$$\begin{cases} u_t + (u^2/2)_x = g(x,t), \\ u(x,0) = \cos(x) \\ u(0,t) = u(2\pi,t) \end{cases} \tag{24}$$

where $g(x,t)$ is given by

$$g(x,t) = -(1+\cos(x+t))\sin(x+t).$$

The exact solution is

$$u(x,t) = \cos(x+t).$$

Table: The errors ξ and e for Example 3 when using P^1 polynomials on a uniform mesh of N cells. $CFL = 0.5$.

P^1	N	$T=1$		$T=50$		$T=500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	20	6.31E-04	–	1.61E-03	–	1.64E-03	–
	40	9.03E-05	2.81	2.74E-04	2.56	2.65E-04	2.63
	80	1.25E-05	2.85	3.76E-05	2.86	4.24E-05	2.65
	160	1.82E-06	2.78	8.15E-06	2.21	6.67E-06	2.67
	320	2.59E-07	2.81	1.50E-06	2.44	1.04E-06	2.68
e	20	4.26E-03	–	4.48E-03	–	4.49E-03	–
	40	1.06E-03	2.00	1.09E-03	2.04	1.09E-03	2.04
	80	2.66E-04	2.00	2.68E-04	2.03	2.69E-04	2.02
	160	6.64E-05	2.00	6.68E-05	2.00	6.67E-05	2.01
	320	1.66E-05	2.00	1.67E-05	2.00	1.66E-05	2.00

Table: The errors ξ and e for Example 3 when using P^2 polynomials on a uniform mesh of N cells. $CFL = 0.5$.

P^2	N	$T = 1$		$T = 50$		$T = 500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	20	7.57E-05	–	9.23E-05	–	1.05E-04	–
	40	8.19E-06	3.21	8.76E-06	3.40	9.08E-06	3.53
	80	9.76E-07	3.07	1.01E-06	3.11	9.11E-07	3.32
	160	8.72E-08	3.48	9.03E-08	3.49	8.81E-08	3.37
e	20	1.20E-04	–	1.31E-04	–	1.31E-04	–
	40	1.47E-05	3.03	1.49E-05	3.13	1.49E-05	3.13
	80	1.77E-06	3.05	1.78E-06	3.07	1.78E-06	3.07
	160	2.15E-07	3.04	2.15E-07	3.04	2.15E-07	3.04

Table: The errors ξ and e for Example 3 when using P^3 polynomials on a uniform mesh of N cells. $CFL = 0.2$.

P^3	N	$T = 1$		$T = 50$		$T = 500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	10	1.10E-05	–	1.56E-05	–	1.50E-05	–
	20	3.94E-07	4.81	4.16E-07	5.23	4.14E-07	5.18
	40	1.49E-08	4.72	1.29E-08	5.01	1.27E-08	5.02
	80	5.39E-10	4.79	3.92E-10	5.04	3.91E-10	5.02
e	10	3.53E-05	–	3.63E-05	–	3.51E-05	–
	20	2.11E-06	4.06	2.11E-06	4.10	2.11E-06	4.05
	40	1.30E-07	4.02	1.30E-07	4.02	1.30E-07	4.02
	80	8.09E-09	4.01	8.08E-09	4.01	8.08E-09	4.01

Example 4: two-dimensional case

Consider

$$\begin{cases} u_t + (u^3/3)_x + (u^3/3)_y = g(x, y, t) \\ u(x, y, 0) = \sin(x + y) \end{cases} \tag{25}$$

where

$$g(x, y, t) = -2\cos^3(x + y - 2t)$$

The exact solution is

$$u(x, y, t) = \sin(x + y - 2t)$$

Table: The errors and orders when using Q^1 and Q^2 polynomials on a nonuniform mesh of $N \times N$ cells. $CFL = 0.2$. $T = 1$

Q^k	$k=1$				$k=2$			
$N \times N$	ξ		e		ξ		e	
	L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
10×10	1.56E-02	–	2.44E-02	–	2.12E-04	–	1.23E-03	–
20×20	2.89E-03	2.57	6.26E-03	2.08	8.89E-06	4.43	1.57E-04	2.87
40×40	5.29E-04	2.53	1.58E-03	2.05	4.89E-07	4.41	2.03E-05	3.11
80×80	9.20E-05	2.61	3.90E-04	2.09	2.79E-08	4.37	2.54E-06	3.18

Summary

- We have proved superconvergence of the DG method for nonlinear hyperbolic conservation laws, under the condition that $|f'(u)|$ has a uniform positive lower bound;
- Numerical experiments are provided to demonstrate the theoretical results.

- Future work
  - Superconvergence of DG method for conservation laws in multidimensional case;
  - Superconvergence property of the local DG (LDG) method for nonlinear diffusion problems.

Thanks for your attention!